Sharpness of the Finsler-Hadwiger inequality

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In the memory of Alexandru Lupaş

1. Introduction & Preliminaries

The Hadwiger-Finsler inequality is known in literature as a generalization of the following

Theorem 1.1. In any triangle ABC with the side lenghts a, b, c and S its area, the following inequality is valid

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3}.$$

This inequality is due to Weitzenbock, Math. Z, 137-146, 1919, but this has also appeared at International Mathematical Olympiad in 1961. In [7.], one can find eleven proofs. In fact, in any triangle ABC the following sequence of inequalities is valid:

$$a^2 + b^2 + c^2 \geq ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \geq 3\sqrt[3]{a^2b^2c^2} \geq 4S\sqrt{3}.$$

A stronger version is the one found by Finsler and Hadwiger in 1938, which states that ([2.])

Theorem 1.2. In any triangle ABC with the side lenghts a, b, c and S its area, the following inequality is valid

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3} + (a-b)^2 + (b-c)^2 + (c-a)^2.$$

In [8.] the first author of this note gave a simple proof only by using AM-GM and the following inequality due to Mitrinovic:

Theorem 1.3. In any triangle ABC with the side lenghts a, b, c and s its semiperimeter and R its circumradius, the following inequality holds

$$s \leq \frac{3\sqrt{3}}{2}R.$$

This inequality also appears in [3.].

A nice inequality, sharper than Mitrinovic and equivalent to the first theorem is the following:

Theorem 1.4. In any triangle ABC with sides of lenghts a, b, c and with inradius of r, circumradius of R and s its semiperimeter the following inequality holds

$$4R + r \geq s\sqrt{3}.$$

In [4.], Wu gave a nice sharpness and a generalization of the Finsler-Hadwiger inequality.

Now, we give an algebraic inequality due to I. Schur ([5.]), namely

Theorem 1.5. For any positive real numbers x, y, z and $t \in \mathbb{R}$ the following inequality holds

$$x^t(x-y)(x-z) + y^t(y-x)(y-z) + z^t(z-y)(z-x) \geq 0.$$

The most common case is $t = 1$, which has the following equivalent form:

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x+y) + yz(y+z) + zx(z+x)$$

which is equivalent to

$$x^3 + y^3 + z^3 + 6xyz \geq (x+y+z)(xy+yz+zx).$$

Now, using the identity $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$ one can easily deduce that

$$2(xy+yz+zx) - (x^2+y^2+z^2) \leq \frac{9xyz}{x+y+z}.(*)$$

Another interesting case is $t = 2$. We have

$$x^4 + y^4 + z^4 + xyz(x+y+z) \geq xy(x^2+y^2) + yz(y^2+z^2) + zx(z^2+x^2)$$

which is equivalent to

$$x^4 + y^4 + z^4 + 2xyz(x+y+z) \geq (x^2+y^2+z^2)(xy+yz+zx).(**)$$

Now, let's rewrite theorem 1.2. as

$$2(ab+bc+ca) - (a^2+b^2+c^2) \geq 4S\sqrt{3}.(***)$$

By squaring $(***)$ and using Heron formula we obtain

$$4\left(\sum_{cyc} ab\right)^2 + \left(\sum_{cyc} a^2\right)^2 - 4\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right) \geq 3(a+b+c)\prod(b+c-a)$$

which is equivalent to

$$6\sum_{cyc} a^2b^2 + 4\sum_{cyc} a^2bc + \sum_{cyc} a^4 - 4\sum_{cyc} ab(a+b) \geq 3(a+b+c)\prod(b+c-a).$$

By making some elementary calculations we get

$$6\sum_{cyc} a^2b^2+4\sum_{cyc} a^2bc+\sum_{cyc} a^4-4\sum_{cyc} ab(a+b) \geq 3(a+b+c)\left(\sum_{cyc} ab(a+b)-\sum_{cyc} a^3-2abc\right).$$

We obtain the equivalent inequalities

$$\sum_{cyc} a^4+\sum_{cyc} a^2bc \geq \sum_{cyc} ab(a^2+b^2)$$

$$a^2(a-b)(a-c)+b^2(b-a)(b-c)+c^2(c-a)(c-b) \geq 0,$$

which is nothing else than Schur's inequality in the particular case $t=2$. In what follows we will give another form of Schur's inequality. That is

Theorem 1.6. For any positive reals m, n, p, the following inequality holds

$$\frac{mn}{p}+\frac{np}{m}+\frac{pm}{n}+\frac{9mnp}{mn+np+pm} \geq 2(m+n+p).$$

Proof. We denote $x=\frac{1}{m}, y=\frac{1}{n}$ and $z=\frac{1}{p}$. We obtain the equivalent inequality

$$\frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy}+\frac{9}{x+y+z} \geq \frac{2(xy+yz+zx)}{xyz} \Leftrightarrow$$

$$2(xy+yz+zx)-(x^2+y^2+z^2) \leq \frac{9xyz}{x+y+z},$$

which is $(*)$.

2. Main results

In the previous section we stated a sequence of inequalities stronger than Weitzenbock inequality. In fact, one can prove that the following sequence of inequalities holds

$$a^2+b^2+c^2 \geq ab+bc+ca \geq a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \geq 3\sqrt[3]{a^2b^2c^2} \geq 18Rr,$$

where R is the circumradius and r is the inradius of the triangle with sides of lenghts a, b, c. In this moment, one expects to have a stronger Finsler-Hadwiger inequality with $18Rr$ instead of $4S\sqrt{3}$. Unfortunately, the following inequality holds true

$$a^2+b^2+c^2 \leq 18Rr+(a-b)^2+(b-c)^2+(c-a)^2,$$

because it is equivalent to

$$2(ab+bc+ca)-(a^2+b^2+c^2) \leq 18Rr=\frac{9abc}{a+b+c},$$

which is $(*)$ again. Now, we are ready to prove the first refinement of the Finsler-Hadwiger inequality:

Theorem 2.1. In any triangle ABC with the side lenghts a, b, c with S its area, R the circumradius and r the inradius of the triangle ABC the following inequality is valid

$$a^2 + b^2 + c^2 \geq 2S\sqrt{3} + 2r(4R + r) + (a - b)^2 + (b - c)^2 + (c - a)^2.$$

Proof. We rewrite the inequality as

$$2(ab + bc + ca) - (a^2 + b^2 + c^2) \geq 2S\sqrt{3} + 2r(4R + r).$$

Since, $ab+bc+ca = s^2+r^2+4Rr$, it follows immediately that $a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$. The inequality is equivalent to

$$16Rr + 4r^2 \geq 2S\sqrt{3} + 2r(4R + r).$$

We finally obtain

$$4R + r \geq s\sqrt{3},$$

which is exactly theorem 1.4.

□

The second refinement of the Finsler-Hadwiger inequality is the following

Theorem 2.2. In any triangle ABC with the side lenghts a, b, c with S its area, R the circumradius and r the inradius of the triangle ABC the following inequality is valid

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3 + \frac{4(R - 2r)}{4R + r}} + (a - b)^2 + (b - c)^2 + (c - a)^2.$$

Proof. In theorem 1.6 we put $m = \frac{1}{2}(b+c-a), n = \frac{1}{2}(c+a-b)$ and $p = \frac{1}{2}(a+b-c)$. We get

$$\sum_{cyc} \frac{(b + c - a)(c + a - b)}{(a + b - c)} + \frac{9(b + c - a)(c + a - b)(a + b - c)}{\sum_{cyc}(b + c - a)(c + a - b)} \geq 2(a + b + c).$$

Since $ab + bc + ca = s^2 + r^2 + 4Rr$ (1) and $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$ (2), we deduce

$$\sum_{cyc}(b + c - a)(c + a - b) = 4r(4R + r).$$

On the other hand, by Heron's formula we have $(b + c - a)(c + a - b)(a + b - c) = 8sr^2$, so our inequality is equivalent to

$$\sum_{cyc} \frac{(b + c - a)(c + a - b)}{(a + b - c)} + \frac{18sr}{4R + r} \geq 4s \Leftrightarrow$$

$$\sum_{cyc} \frac{(s-a)(s-b)}{(s-c)} + \frac{9sr}{4R+r} \geq 2s \Leftrightarrow \sum_{cyc} (s-a)^2(s-b)^2 + \frac{9s^2r^3}{4R+r} \geq 2s^2r^2.$$

Now, according to the identity

$$\sum_{cyc} (s-a)^2(s-b)^2 = \left(\sum_{cyc} (s-a)(s-b)\right)^2 - 2s^2r^2,$$

we have

$$\left(\sum_{cyc} (s-a)(s-b)\right)^2 - 2s^2r^2 + \frac{9s^2r^3}{4R+r} \geq 2s^2r^2.$$

And since

$$\sum_{cyc} (s-a)(s-b) = r(4R+r),$$

it follows that

$$r^2(4R+r)^2 + \frac{9s^2r^3}{4R+r} \geq 4s^2r^2,$$

which rewrites as

$$\left(\frac{4R+r}{s}\right)^2 + \frac{9r}{4R+r} \geq 4.$$

From the identities mentioned in (1) and (2) we deduce that

$$\frac{4R+r}{s} = \frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{4S}.$$

The inequality rewrites as

$$\left(\frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{4S}\right)^2 \geq 4 - \frac{9r}{4R+r} \Leftrightarrow$$

$$\left(\frac{(a^2+b^2+c^2)-\left((a-b)^2+(b-c)^2+(c-a)^2\right)}{4S}\right)^2 \geq 3 + \frac{4(R-2r)}{4R+r} \Leftrightarrow$$

$$a^2+b^2+c^2 \geq 4S\sqrt{3+\frac{4(R-2r)}{4R+r}} + (a-b)^2+(b-c)^2+(c-a)^2.$$

□

Remark. From Euler inequality, $R \geq 2r$, we obtain Theorem 1.2.

3. Applications

In this section we illustrate some basic applications of the second refinement of Finsler-Hadwiger inequality. We begin with

Problem 1. *In any triangle ABC with the sides of lenghts a, b, c the following inequality holds*

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{c+a-b} \geq \frac{1}{2r}\sqrt{4-\frac{9r}{4R+r}}.$$

*Solution.*From

$$(b+c-a)(c+a-b)(a+b-c)=4r(4R+r),$$

it is quite easy to observe that

$$\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}=\frac{4R+r}{2sr}.$$

Now, applying the inequality

$$\left(\frac{4R+r}{s}\right)^2+\frac{9r}{4R+r} \geq 4,$$

we get

$$\left(\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}\right)^2=\frac{1}{4r^2}\left(\frac{4R+r}{s}\right)^2 \geq \frac{1}{4r^2}\left(4-\frac{9r}{4R+r}\right).$$

The given inequality follows immediately. □

Problem 2. *In any triangle ABC with the sides of lenghts a, b, c the following inequality holds*

$$\frac{1}{a(b+c-a)}+\frac{1}{b(c+a-b)}+\frac{1}{c(a+b-c)} \geq \frac{r}{8R}\left(5-\frac{9r}{4R+r}\right).$$

*Solution.*From the following identity

$$\sum_{cyc}\frac{(s-a)(s-b)}{c}=\frac{r(s^2+(4R+r)^2)}{4sR}=\frac{S}{4R}\left(1+\left(\frac{4R+r}{p}\right)^2\right).$$

Using the inequality

$$\left(\frac{4R+r}{s}\right)^2+\frac{9r}{4R+r} \geq 4,$$

we have

$$\sum_{cyc}\frac{(s-a)(s-b)}{c} \geq \frac{S}{4R}\left(5-\frac{9r}{4R+r}\right).$$

In this moment, the problem follows easily. □

Problem 3. *In any triangle ABC with the sides of lenghts a, b, c the following inequality holds*

$$\frac{1}{(b+c-a)^2}+\frac{1}{(c+a-b)^2}+\frac{1}{(a+b-c)^2} \geq \frac{1}{r^2}\left(\frac{1}{2}-\frac{9r}{4(4R+r)}\right).$$

*Solution.*From $(b+c-a)(c+a-b)(a+b-c) = 4r(4R+r)$, it follows that

$$(b+c-a)^2 + (c+a-b)^2 + (a+b-c)^2 = 4(s^2 - 2r^2 - 8Rr)$$

and

$$(b+c-a)^2(c+a-b)^2 + (a+b-c)^2(c+a-b)^2 + (b+c-a)^2(a+b-c)^2 = 4r^2\left((4R+r)^2 - 2s^2\right).$$

We get

$$\frac{1}{(b+c-a)^2} + \frac{1}{(c+a-b)^2} + \frac{1}{(a+b-c)^2} = \frac{1}{4}\left(\frac{(4R+r)^2}{s^2r^2} - \frac{2}{r^2}\right).$$

Now, applying the inequality

$$\left(\frac{4R+r}{s}\right)^2 + \frac{9r}{4R+r} \geq 4,$$

we have

$$\frac{1}{(b+c-a)^2} + \frac{1}{(c+a-b)^2} + \frac{1}{(a+b-c)^2} \geq \frac{1}{4r^2}\left(2 - \frac{9r}{4R+r}\right) = \frac{1}{r^2}\left(\frac{1}{2} - \frac{9r}{4(4R+r)}\right).$$

□

Problem 4. *In any triangle ABC with the sides of lenghts a, b, c the following inequality holds*

$$\frac{a^2}{b+c-a} + \frac{b^2}{c+a-b} + \frac{c^2}{a+b-c} \geq 3R\sqrt{4 - \frac{9r}{4R+r}}.$$

*Solution.*Without loss of generality, we assume that $a \leq b \leq c$. It follows quite easily that $a^2 \leq b^2 \leq c^2$ and $\frac{1}{b+c-a} \leq \frac{1}{c+a-b} \leq \frac{1}{a+b-c}$. Applying Chebyshev's inequality, we have

$$\frac{a^2}{b+c-a} + \frac{b^2}{c+a-b} + \frac{c^2}{a+b-c} \geq \left(\frac{a^2+b^2+c^2}{3}\right)\left(\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{c+a-b}\right).$$

Now, the first application and the inequality $a^2 + b^2 + c^2 \geq 18Rr$ solves the problem. □

Remark. Using Euler's inequality, $R \geq 2r$, in the original problem, we obtain the weaker version

$$\frac{a^2}{b+c-a} + \frac{b^2}{c+a-b} + \frac{c^2}{a+b-c} \geq 3R\sqrt{3},$$

which represents an old proposal of Laurenţiu Panaitopol at the Romanian IMO Team Selection Test, held in 1990.

Problem 5. *In any triangle ABC with the sides of lenghts a, b, c and with the exradii r_a, r_b, r_c corresponding to the triangle ABC, the following inequality holds*

$$\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} \geq 2\sqrt{3 + \frac{4(R-2r)}{4R+r}}.$$

*Solution.*From the well-known relations $r_a = \dfrac{S}{s-a}$ and the analogues, the inequality is equivalent to

$$\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} = \frac{2(ab+bc+ca)-(a^2+b^2+c^2)}{2S} \geq 2S\sqrt{3 + \frac{4(R-2r)}{4R+r}}.$$

The last inequality follows from Theorem 2.2. □

Problem 6. *In any triangle ABC with the sides of lenghts a, b, c and with the exradii r_a, r_b, r_c corresponding to the triangle ABC and with h_a, h_b, h_c be the altitudes of the triangle ABC, the following inequality holds*

$$\frac{1}{h_a r_a} + \frac{1}{h_b r_b} + \frac{1}{h_c r_c} \geq \frac{1}{S}\sqrt{3 + \frac{4(R-2r)}{4R+r}}.$$

*Solution.*From the well-known relations in triangle ABC, $h_a = \dfrac{2S}{a}, r_a = \dfrac{S}{s-a}$ we have $\dfrac{1}{h_a r_a} = \dfrac{a(s-a)}{2S^2}$. Doing the same thing for the analogues and adding them up we get

$$\frac{1}{h_a r_a} + \frac{1}{h_b r_b} + \frac{1}{h_c r_c} = \frac{1}{2S^2}\left(a(s-a) + b(s-b) + c(s-c)\right).$$

On the other hand by using Theorem 2.2, in the form

$$a(s-a) + b(s-b) + c(s-c) \geq 2S\sqrt{3 + \frac{4(R-2r)}{4R+r}},$$

we obtain the desired inequality. □

Problem 7. *In any triangle ABC with the sides of lenghts a, b, c the following inequality holds true*

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} \geq \sqrt{3 + \frac{4(R-2r)}{4R+r}}.$$

*Solution.*From the cosine law we get $a^2 = b^2 + c^2 - 2bc\cos A$. Since $S = \frac{1}{2}bc\sin A$ it follows that

$$a^2 = (b-c)^2 + 4S \cdot \frac{1-\cos A}{\sin A}.$$

On the other hand by the trigonometric formulae $1 - \cos A = 2\sin^2\dfrac{A}{2}$ and $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2}$ we get

$$a^2 = (b-c)^2 + 4S\tan\frac{A}{2}.$$

Doing the same for all sides of the triangle ABC and adding up we obtain

$$a^2+b^2+c^2=(a-b)^2+(b-c)^2+(c-a)^2+4S\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right).$$

Now, by Theorem 2.2, the inequality follows. □

Problem 8. *In any triangle ABC with the sides of lenghts a,b,c and with the exradii r_a,r_b,r_c corresponding to the triangle ABC, the following inequality holds*

$$\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}\geq\frac{s(5R-r)}{R(4R+r)}.$$

*Solution.*It is well-known that the following identity is valid in any triangle ABC

$$\frac{r_a}{a}+\frac{r_b}{b}+\frac{r_c}{c}=\frac{(4R+r)^2+s^2}{4Rs}.$$

So, the inequality rewrites as

$$\frac{(4R+r)^2}{s^2}+1\geq\frac{4(5R-r)}{4R+r},$$

which is equivalent with

$$\left(\frac{4R+r}{s}\right)^2+\frac{9r}{4R+r}\geq 4.$$

□

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